Monadic second order logic as a model companion

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Automata and logic: example

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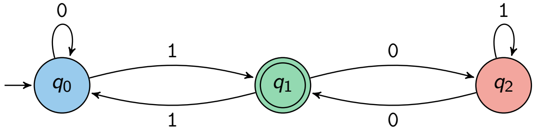

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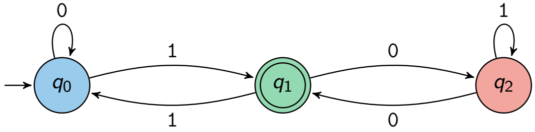

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- Solution 2: a monadic second order formula $\varphi(W_0, W_1)$:

$$\exists Q_0 \exists Q_1 \exists Q_2 (Q_0(\texttt{first}) \wedge Q_1(\texttt{last}) \wedge \text{Partition}(Q_0, Q_1, Q_2) \wedge$$

$$\forall x([W_0(x) \wedge Q_0(x) \to Q_0(\texttt{S}x)] \wedge [W_1(x) \wedge Q_0(x) \to Q_1(\texttt{S}x)] \wedge \ldots))$$

Answer yes iff $w = (W_0, W_1)$ makes φ true.

Regular languages

Regular languages over a finite alphabet Σ are subsets $L \subseteq \Sigma^\omega$ which are ...

- recognizable by a finite automaton;

  or, equivalently,

- definable by a formula of S1S,
  the monadic second order logic of one successor.

Büchi 1960

A model complete theory

A functional language $\mathcal{L}$: Boolean algebra operations $(\bot, \cup, -)$, two unary functions, $\mathbf{X}$ and $\mathbf{F}$, and a constant $\mathbf{I}$.

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The Boolean algebra $\mathcal{P}(\omega)$ is an $\mathcal{L}$-structure with:

- $\mathbf{X}a := \{t \in \omega \mid t + 1 \in a\}$,
- $\mathbf{F}a := \{t \in \omega \mid \exists t' \geq t \colon t' \in a\}$,
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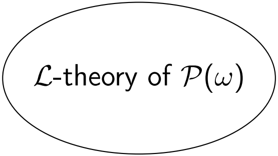
Theorem

The first order $\mathcal{L}$-theory of $\mathcal{P}(\omega)$ is model complete.

A theory T^* is model complete iff every formula is T^*-equivalent to an existential formula.

Ghilardi, G. JSL 2017

Proving model completeness with automata



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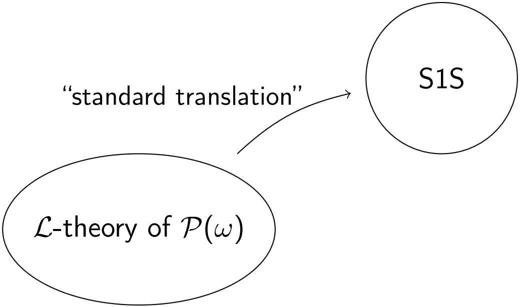

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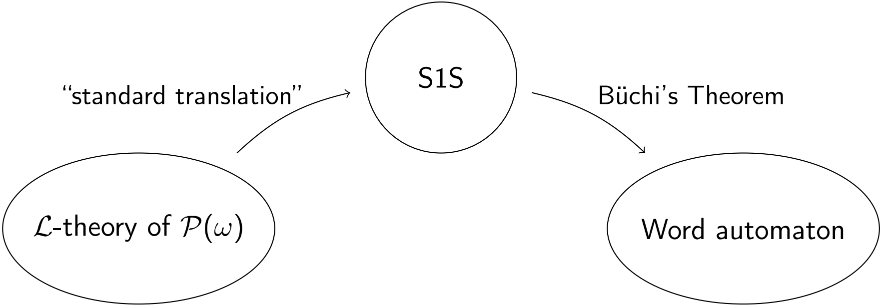

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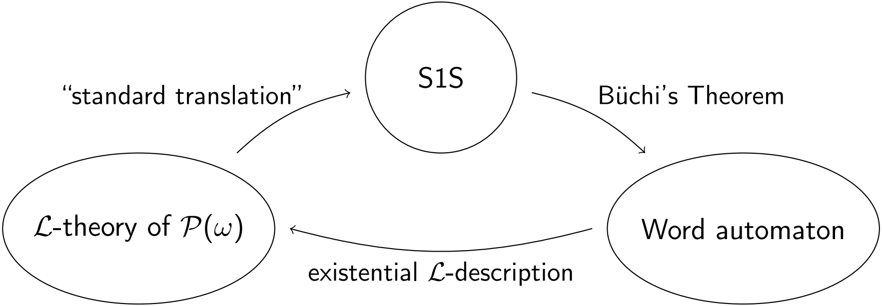

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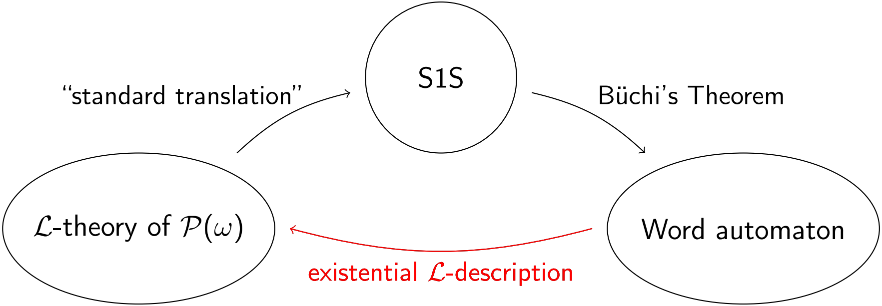

An existential $\mathcal{L}$-description of a word automaton

- Let $A = (Q, \Sigma, \delta, q_0, F)$ be a **word automaton** over a finite alphabet Σ, i.e., a function $\delta\colon Q \times \Sigma \to \mathcal{P}(Q)$, an initial state $q_0 \in Q$ and a subset $F \subseteq Q$ of final states.

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- Write $\Sigma = \{0, \ldots, s\}$, $Q = \{0, \ldots, m\}$, $q_0 = 0$.
- A word $W\colon \omega \to \Sigma$ is a partition $(W_0, \ldots, W_s)$ of ω; $W_j = W^{-1}(j)$.

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Key Observation. The automaton A accepts a word $W\colon \omega \to \Sigma$ iff $\mathcal{P}(\omega), [w_i \mapsto W_i] \models \alpha(w_0, \ldots, w_s)$, where α is the $\exists$ $\mathcal{L}$-formula:

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$$\exists q_0, \ldots, q_m(\text{“the } q_i \text{ partition } \omega\text{”} \wedge \bigwedge_{\substack{0 \le i \le m \\ 0 \le j \le s}} \left(q_i \cap w_j \subseteq \bigcup_{k \in \delta(i,j)} \mathbf{X} q_k \right)$$

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$$\wedge\ I \subseteq q_0 \wedge \mathbf{F} \left(\textstyle\bigcup_{i \in F} q_i \right) = \top).$$

The theory is a model companion

A theory T^* is a model companion of a theory T iff T^* is model complete, and T and T^* have the same universal consequences.

Theorem

The $\mathcal{L}$-theory of $\mathcal{P}(\omega)$ is the model companion of the theory of $\mathcal{L}$-structures axiomatized by the following universal sentences:

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Theorem

The $\mathcal{L}$-theory of $\mathcal{P}(\omega)$ is the model companion of the theory of $\mathcal{L}$-structures axiomatized by the following universal sentences:

- *equations for Boolean algebras;*
- $\mathbf{X}$ *is a Boolean homomorphism;*
- $\mathbf{F}a$ *is the least fixed point of* $x \mapsto a \vee \mathbf{X}x$*;*
- $\mathbf{I}$ *is an atom which is below* $\mathbf{F}a$ *for any* $a \neq \bot$*, and* $\mathbf{X}\mathbf{I} = \bot$*.*

Ghilardi, G. JSL 2017

Binary trees

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The Boolean algebra $\mathcal{P}(2^*)$ is an $\mathcal{L}_2$-structure with

- $\mathbf{I} := \{\epsilon\}$,
- $\mathbf{X}_i a := \{t \in \omega \mid t \cdot i \in a\}$ for $i = 0, 1$,
- $t \in \mathbf{EU}(a, b)$ iff there **E**xists a path $t = t_0, \ldots, t_n$ such that, for $i < n$, $t_i \in a$, and (**U**ntil) $t_n \in b$,

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- $t \in \mathbf{AF}(a, -b)$ iff for **A**ll infinite paths $t = t_0, t_1, \ldots$ there is a (**F**uture) $t_i \in a$, provided that $t_j \in b$ for infinitely many j.

Model companion for binary trees

Theorem

The $\mathcal{L}_2$-theory of $\mathcal{P}(2^*)$ is model complete, and is in fact the model companion of an $\mathcal{L}_2$-theory with a finite universal axiomatization.

Ghilardi, G. LICS 2016

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- Proving model completeness crucially uses tree automata originally developed for deciding S2S (Rabin 1969).
- We obtain an analogous result for 'bisimulation-invariant' MSO, i.e., the modal μ-calculus (Janin-Walukiewicz 1996).

Ongoing work and questions

- Ongoing work: extending these results to general trees; this requires an infinite language that can count successors.
- Where do $\mathcal{L}$-structures and $\mathcal{L}_2$-structures fit in model theory?
  - Context: model companions also exist for Heyting algebras and certain modal algebras; but the methods are different.
- Can automata methods be useful for proving the model completeness of other theories (especially if they have a 'computation' flavor)?

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