

Towards an independent version of Tarski's system of geometry

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Introduction

Introduction

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- Hilbert's *Grundlagen der Geometrie*

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Introduction

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- Hilbert's *Grundlagen der Geometrie* contains a chapter dedicated to independence properties.
- Tarski's System of Geometry and the problem of its independence was carefully studied by Gupta.

Outline

- 1 Introduction
- 2 Tarski's System of Geometry
- 3 Gupta's and Szczerba's contributions
- 4 An independent version of Tarski's system of geometry?
- 5 Conclusion

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Introduction

Tarski's System of Geometry

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The axioms

A model of the theory

Tarski's system of geometry

Tarski's system of geometry

- A single primitive type: point.



Alfred Tarski
(1901 - 1983)

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- 11 axioms.



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- 11 axioms.
- A parameter controls the dimension.



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- A single primitive type: point.
- Two primitive predicates:
 - ① congruence $AB \equiv CD$;
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- 11 axioms.
- A parameter controls the dimension.
- Good meta-theoretical properties.



Alfred Tarski
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Axioms about congruence

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Axiom (Pseudo-transitivity for congruence)

$$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$$

Axioms about congruence

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Axiom (Identity for congruence)

$$AB \equiv CC \Rightarrow A = B$$

Axiom about betweenness

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Axiom (Identity for betweenness)

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Five-Segment Axiom

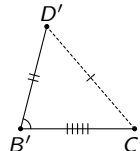
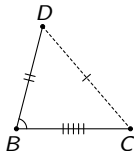
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Axiom (Five-Segment)

$$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$$

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$$A-B-C \wedge A'-B'-C' \wedge A \neq B \Rightarrow CD \equiv C'D'$$



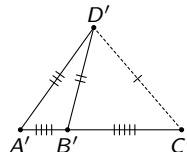
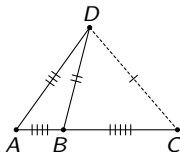
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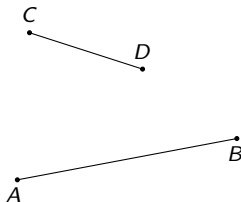
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Axiom (Segment Construction)

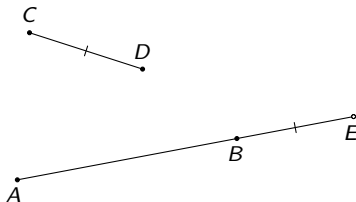
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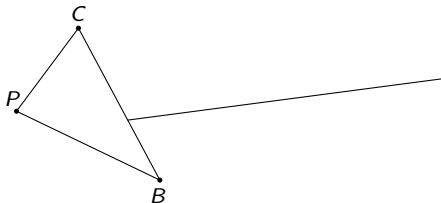


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Axiom (Pasch)

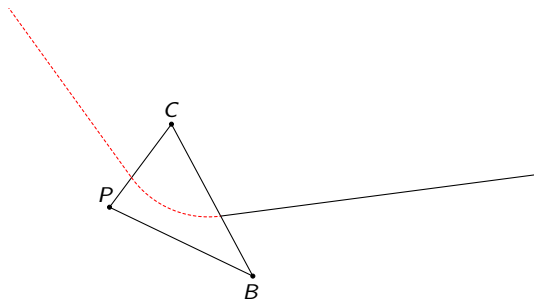
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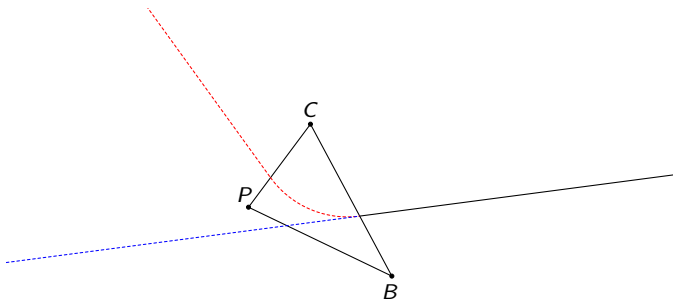
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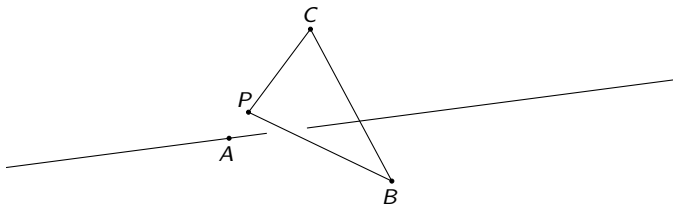
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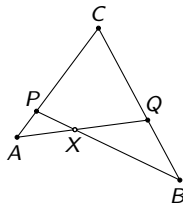
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2-Dimensional Axiom

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Axiom (Lower 2-Dimensional)

$$\exists ABC, \neg A-B-C \wedge \neg B-C-A \wedge \neg C-A-B$$

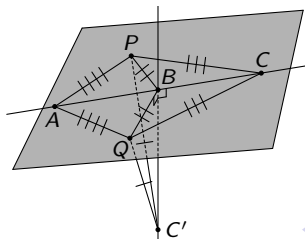
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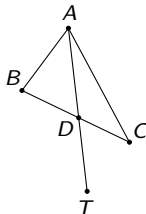


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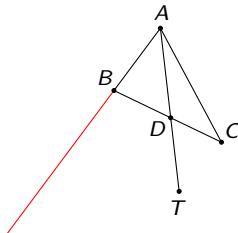
$$A-D-T \wedge B-D-C \wedge A \neq D \Rightarrow \\ \exists XY, A-B-X \wedge A-C-Y \wedge X-T-Y$$



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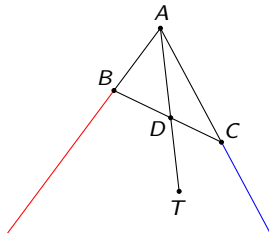
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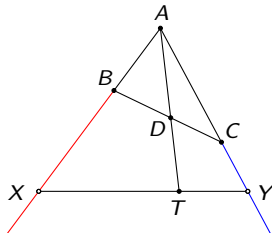
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Identity for betweenness	$A-B-A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A-B-E \wedge BE \equiv CD$
Pasch	$A-P-C \wedge B-Q-C \Rightarrow \exists X, P-X-B \wedge Q-X-A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A-B-C \wedge A'-B'-C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A-B-C \wedge \neg B-C-A \wedge \neg C-A-B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A-B-C \vee B-C-A \vee C-A-B$
Euclid	$A-D-T \wedge B-D-C \wedge A \neq D \Rightarrow$ $\exists XY, A-B-X \wedge A-C-Y \wedge X-T-Y$
Continuity	$\forall \exists \Upsilon, (\exists A, (\forall XY, \exists X \wedge \Upsilon Y \Rightarrow A-X-Y)) \Rightarrow$ $\exists B, (\forall XY, \exists X \wedge \Upsilon Y \Rightarrow X-B-Y)$

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Point equality decidability	$X = Y \vee X \neq Y$

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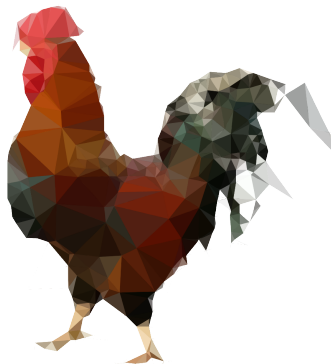
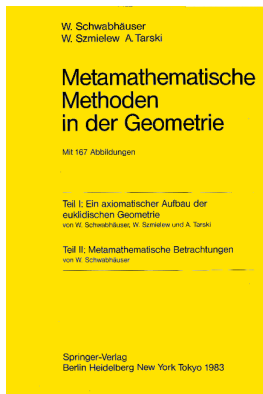
Conclusion

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A model of the theory

Overview of the formalization

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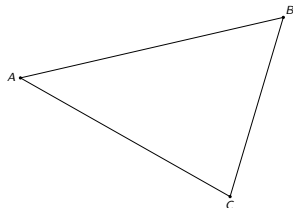


geocoq.github.io/GeoCoq/

An example of proof by computation

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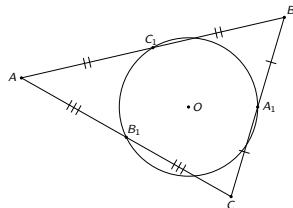
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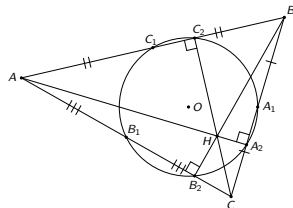
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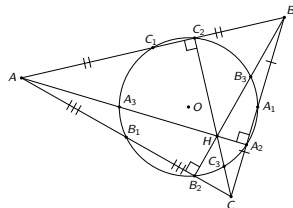
- The midpoints of each side of the triangle;
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An example of proof by computation

Our example is the nine-point circle theorem which states that the following nine points are concyclic:

- The midpoints of each side of the triangle;
- The feet of each altitude;
- The midpoints of the line-segments from each vertex of the triangle to the orthocenter.



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- $AB \equiv CD := (x_A - x_B)^2 + (y_A - y_B)^2 = (x_C - x_D)^2 + (y_C - y_D)^2$.

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- Points: \mathbb{F}^2 where \mathbb{F} is a real closed field.
- $AB \equiv CD := (x_A - x_B)^2 + (y_A - y_B)^2 = (x_C - x_D)^2 + (y_C - y_D)^2$.
- $A-B-C := \exists k, 0 \leq k \leq 1 \wedge B - A = k(C - A)$.

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Gupta's contribution

Szczerba's contribution

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Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A-B-C \wedge A'-B'-C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A-B-C \wedge \neg B-C-A \wedge \neg C-A-B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A-B-C \vee B-C-A \vee C-A-B$
Euclid	$A-D-T \wedge B-D-C \wedge A \neq D \Rightarrow$ $\exists XY, A-B-X \wedge A-C-Y \wedge X-T-Y$
Continuity	$\forall \exists \Upsilon, (\exists A, (\forall XY, \exists X \wedge \Upsilon Y \Rightarrow A-X-Y)) \Rightarrow$ $\exists B, (\forall XY, \exists X \wedge \Upsilon Y \Rightarrow X-B-Y)$

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Lower 2-Dimensional

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$$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$$

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Lower 2-Dimensional	$\exists ABC, \neg A-B-C \wedge \neg B-C-A \wedge \neg C-A-B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A-B-C \vee B-C-A \vee C-A-B$
Euclid	$A-D-T \wedge B-D-C \wedge A \neq D \Rightarrow$ $\exists XY, A-B-X \wedge A-C-Y \wedge X-T-Y$
Continuity	$\forall \exists \Upsilon, (\exists A, (\forall XY, \exists X \wedge \Upsilon Y \Rightarrow A-X-Y)) \Rightarrow$ $\exists B, (\forall XY, \exists X \wedge \Upsilon Y \Rightarrow$ $X-B-Y \vee X \neq B \vee B = Y)$

Gupta's axiom system

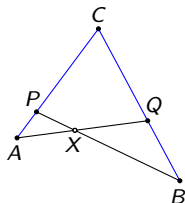
Transitivity for betweenness	$A-B-D \wedge B-C-D \Rightarrow A-B-C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A-B-E \wedge BE \equiv CD$
Pasch	$A-X-Q \wedge C-Q-B \Rightarrow \exists P, A-P-C \wedge B-X-P$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A-B-C \wedge A'-B'-C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A-B-C \wedge \neg B-C-A \wedge \neg C-A-B$
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Pasch's axioms

Pasch's axioms

Axiom (Inner Pasch)

$$A-P-C \wedge B-Q-C \Rightarrow \exists X, P-X-B \wedge Q-X-A$$



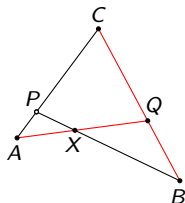
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$$A-X-Q \wedge C-Q-B \Rightarrow \exists P, A-P-C \wedge B-X-P$$



Introduction

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Gupta's contribution

Szczerba's contribution

Gupta's independence model for Pasch

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- Points: \mathbb{F}^2 where \mathbb{F} is a real closed field.

Gupta's independence model for Pasch

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- $AB \equiv CD := (x_A - x_B)^2 + (y_A - y_B)^2 = (x_C - x_D)^2 + (y_C - y_D)^2$.

Gupta's independence model for Pasch

- Points: \mathbb{F}^2 where \mathbb{F} is a real closed field.
- $AB \equiv CD := (x_A - x_B)^2 + (y_A - y_B)^2 = (x_C - x_D)^2 + (y_C - y_D)^2$.
- $A-B-C := \exists k, 0 \leq k \leq 1 \wedge B - A = k(C - A)$ at the exception of the cases when $A = B$ and both A and C belong to the x-axis.

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Szczerba's axiom system

Szczerba's axiom system

Identity for betweenness

Transitivity for congruence

Reflexivity for congruence

Identity for congruence

Segment Construction

Pasch $A-X-Q \wedge C-Q-B \Rightarrow \exists P, A-P-C \wedge B-X-P$

Five-Segment

Lower 2-Dimensional

Upper 2-Dimensional

Euclid $\neg(A-B-C \vee B-C-A \vee C-A-B) \Rightarrow$
 $\exists C_C, AC_C \equiv BC_C \wedge AC_C \equiv CC_C$

Continuity

Szczerba's axiom system

Identity for betweenness

Transitivity for congruence

Reflexivity for congruence

Identity for congruence

Segment Construction

Pasch

$$A-X-Q \wedge C-Q-B \Rightarrow \exists P, A-P-C \wedge B-X-P$$

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Lower 2-Dimensional

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Triangle circumscription principle

Triangle circumscription principle

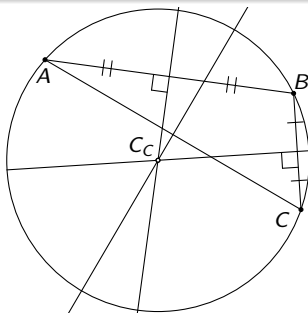
Axiom (Triangle circumscription principle)

$$\neg(A-B-C \vee B-C-A \vee C-A-B) \Rightarrow \\ \exists C_C, AC_C \equiv BC_C \wedge AC_C \equiv CC_C$$

Triangle circumscription principle

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- 3 Gupta's and Szczerba's contributions
- 4 An independent version of Tarski's system of geometry?
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Introduction

Tarski's System of Geometry

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Conclusion

Parallel postulates are not *equivalent*

How to classify the postulates?

The axioms

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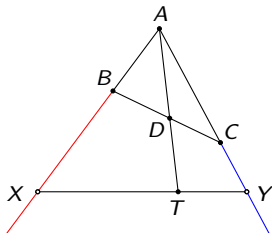
The axioms

Euclid's axiom

Euclid's axiom

Axiom (Euclid)

$$A-D-T \wedge B-D-C \wedge A \neq D \Rightarrow \\ \exists XY, A-B-X \wedge A-C-Y \wedge X-T-Y$$



Syntactic proof

Identity for betweenness	$A-B-A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A-B-E \wedge BE \equiv CD$
Pasch	$A-P-C \wedge B-Q-C \Rightarrow \exists X, P-X-B \wedge Q-X-A$
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Lower 2-Dimensional	$\exists ABC, \neg A-B-C \wedge \neg B-C-A \wedge \neg C-A-B$
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Continuity	$\forall \exists \Upsilon, (\exists A, (\forall XY, X \in \Xi \wedge Y \in \Upsilon \Rightarrow A-X-Y)) \Rightarrow$ $\exists B, (\forall XY, X \in \Xi \wedge Y \in \Upsilon \Rightarrow X-B-Y)$

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Syntactic proof

$$A-B-A \Rightarrow A = B$$

$$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$$

$$AB \equiv BA$$

$$AB \equiv CC \Rightarrow A = B$$

$$\exists E, A-B-E \wedge BE \equiv CD$$

$$A-P-C \wedge B-Q-C \Rightarrow \exists X, P-X-B \wedge Q-X-A$$

$$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$$

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$$\exists ABC, \neg A-B-C \wedge \neg B-C-A \wedge \neg C-A-B$$

$$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$$

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Syntactic proof

$B \circ$

$\circ A$

$\circ C$

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$$\exists ABC, \neg A-B-C \wedge \neg B-C-A \wedge \neg C-A-B$$

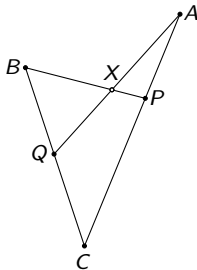
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Syntactic proof



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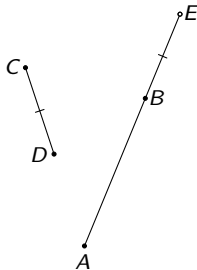
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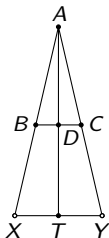
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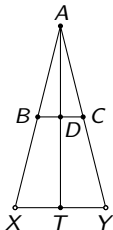
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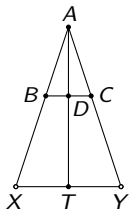
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$$\exists ABC, \neg A-B-C \wedge \neg B-C-A \wedge \neg C-A-B$$

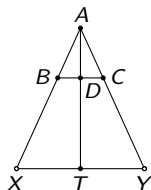
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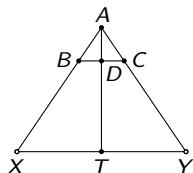
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Syntactic proof



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Identity for betweenness	$A-B-A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A-B-E \wedge BE \equiv CD$
Pasch	$A-P-C \wedge B-Q-C \Rightarrow \exists X, P-X-B \wedge Q-X-A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A-B-C \wedge A'-B'-C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A-B-C \wedge \neg B-C-A \wedge \neg C-A-B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A-B-C \vee B-C-A \vee C-A-B$
Euclid	$A-D-T \wedge B-D-C \wedge A \neq D \Rightarrow$ $\exists XY, A-B-X \wedge A-C-Y \wedge X-T-Y$
Continuity	$\forall \exists \Upsilon, (\exists A, (\forall XY, X \in \Xi \wedge Y \in \Upsilon \Rightarrow A-X-Y)) \Rightarrow$ $\exists B, (\forall XY, X \in \Xi \wedge Y \in \Upsilon \Rightarrow X-B-Y)$

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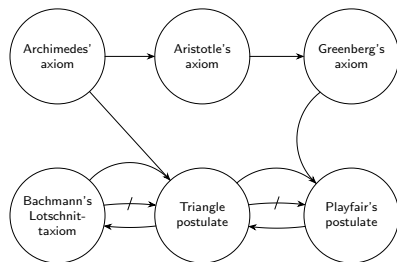
Parallel postulates are not *equivalent*

How to classify the postulates?

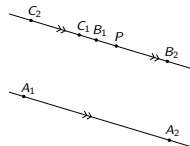
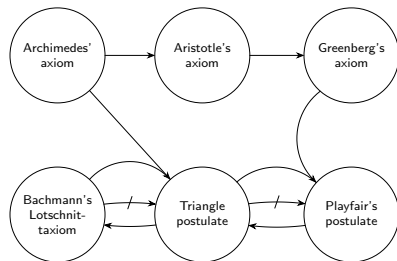
The axioms

Parallel postulates are not *equivalent*

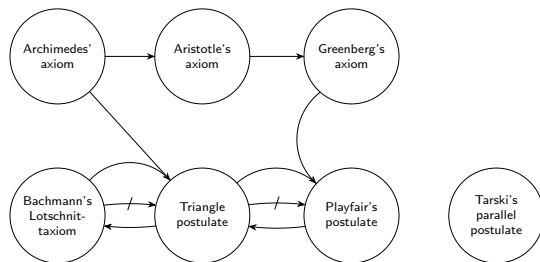
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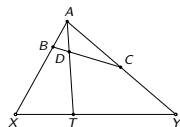
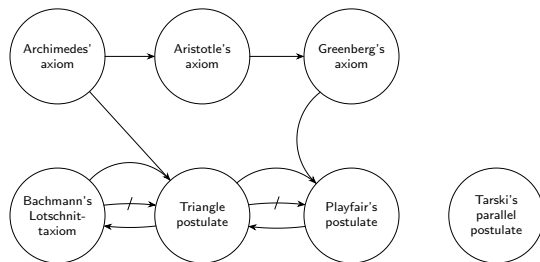
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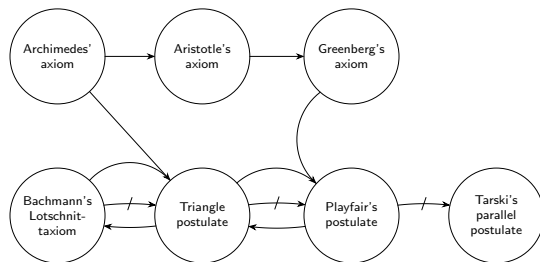
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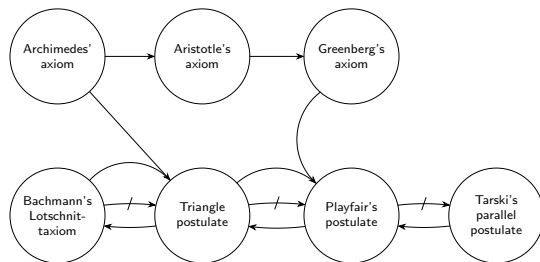
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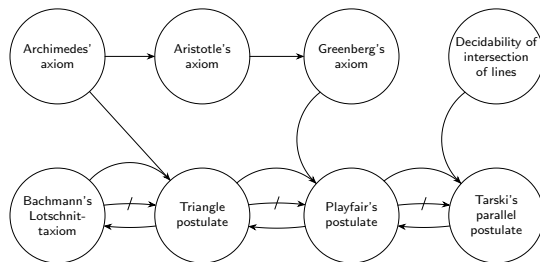
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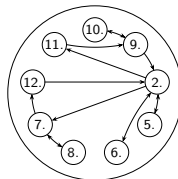
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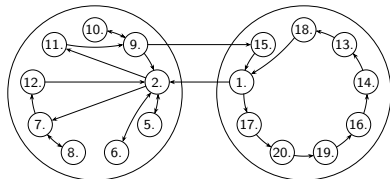
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How to classify the postulates?

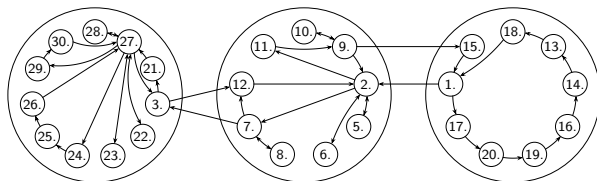
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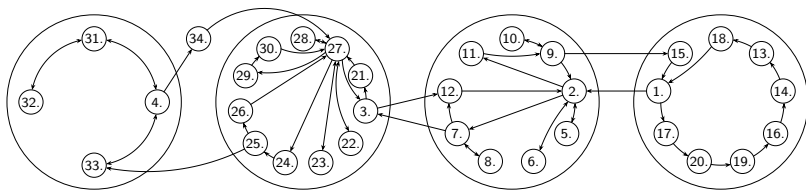
How to classify the postulates?



How to classify the postulates?



How to classify the postulates?



How to classify the postulates?

Pursuing the project faithfully will require that we take the extreme measure of shutting out the entreaties of our intuitions and imaginations - a forced separation of mental powers that will quite understandably be confusing and difficult to maintain [...].

(Richard J. Trudeau)

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The axioms

Transitivity for betweenness	$A-B-D \wedge B-C-D \Rightarrow A-B-C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A-B-E \wedge BE \equiv CD$
Pasch	$A-X-Q \wedge C-Q-B \Rightarrow \exists P, A-P-C \wedge B-X-P$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A-B-C \wedge A'-B'-C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A-B-C \wedge \neg B-C-A \wedge \neg C-A-B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A-B-C \vee B-C-A \vee C-A-B$
Euclid	$A-D-T \wedge B-D-C \wedge A \neq D \Rightarrow$ $\exists XY, A-B-X \wedge A-C-Y \wedge X-T-Y$
Continuity	$\forall \exists T, (\exists A, (\forall XY, \exists X \wedge T Y \Rightarrow A-X-Y)) \Rightarrow$ $\exists B, (\forall XY, \exists X \wedge T Y \Rightarrow X-B-Y \vee X = B \vee B = Y)$

The axioms

Symmetry for betweenness

$$A-B-C \Rightarrow C-B-A$$

Transitivity for betweenness

$$A-B-D \wedge B-C-D \Rightarrow A-B-C$$

Transitivity for congruence

$$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$$

Reflexivity for congruence

$$AB \equiv BA$$

Identity for congruence

$$AB \equiv CC \Rightarrow A = B$$

Segment Construction

$$\exists E, A-B-E \wedge BE \equiv CD$$

Pasch

$$A-X-Q \wedge C-Q-B \Rightarrow \exists P, A-P-C \wedge B-X-P$$

Five-Segment

$$\begin{aligned} AB \equiv A'B' \wedge BC \equiv B'C' \wedge \\ AD \equiv A'D' \wedge BD \equiv B'D' \wedge \\ A-B-C \wedge A'-B'-C' \wedge A \neq B \Rightarrow CD \equiv C'D' \end{aligned}$$

Lower 2-Dimensional

$$\exists ABC, \neg A-B-C \wedge \neg B-C-A \wedge \neg C-A-B$$

Upper 2-Dimensional

$$\begin{aligned} AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge \\ A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow \\ A-B-C \vee B-C-A \vee C-A-B \end{aligned}$$

Euclid

$$\begin{aligned} A-D-T \wedge B-D-C \wedge A \neq D \Rightarrow \\ \exists XY, A-B-X \wedge A-C-Y \wedge X-T-Y \end{aligned}$$

Continuity

$$\begin{aligned} \forall \exists T, (\exists A, (\forall XY, \exists X \wedge T Y \Rightarrow A-X-Y)) \Rightarrow \\ \exists B, (\forall XY, \exists X \wedge T Y \Rightarrow X-B-Y \vee X=B \vee B=Y) \end{aligned}$$

The axioms

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Point equality decidability	$X = Y \vee X \neq Y$

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Point equality decidability	$X = Y \vee X \neq Y$

The axioms

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Lower 2-Dimensional	$\exists ABC, \neg A-B-C \wedge \neg B-C-A \wedge \neg C-A-B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A-B-C \vee B-C-A \vee C-A-B$

Proclus

Continuity	$\forall \exists \Upsilon, (\exists A, (\forall XY, \exists X \wedge \Upsilon Y \Rightarrow A-X-Y)) \Rightarrow$ $\exists B, (\forall XY, \exists X \wedge \Upsilon Y \Rightarrow X-B-Y \vee X=B \vee B=Y)$
Point equality decidability	$X=Y \vee X \neq Y$

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A few definitions

Collinearity

$$A-B-C \vee B-C-A \vee C-A-B$$

A few definitions

Collinearity

$$A-B-C \vee B-C-A \vee C-A-B$$

Coplanarity

$$\exists X, (Col\ A\ B\ X \wedge Col\ C\ D\ X) \vee (Col\ A\ C\ X \wedge Col\ B\ D\ X) \vee \\ (Col\ A\ D\ X \wedge Col\ B\ C\ X)$$

A few definitions

Collinearity

$$A-B-C \vee B-C-A \vee C-A-B$$

Coplanarity

$$\exists X, (\text{Col } A B X \wedge \text{Col } C D X) \vee (\text{Col } A C X \wedge \text{Col } B D X) \vee (\text{Col } A D X \wedge \text{Col } B C X)$$

Strict parallelism

$$A \neq B \wedge C \neq D \wedge \text{Cp } A B C D \wedge \neg \exists X, \text{Col } A B X \wedge \text{Col } C D X$$

A few definitions

Collinearity

$$A-B-C \vee B-C-A \vee C-A-B$$

Coplanarity

$$\exists X, (\text{Col } A B X \wedge \text{Col } C D X) \vee (\text{Col } A C X \wedge \text{Col } B D X) \vee (\text{Col } A D X \wedge \text{Col } B C X)$$

Strict parallelism

$$A \neq B \wedge C \neq D \wedge \text{Cp } A B C D \wedge \neg \exists X, \text{Col } A B X \wedge \text{Col } C D X$$

Parallelism

$$AB \parallel_s CD \vee (A \neq B \wedge C \neq D \wedge \text{Col } A C D \wedge \text{Col } B C D)$$

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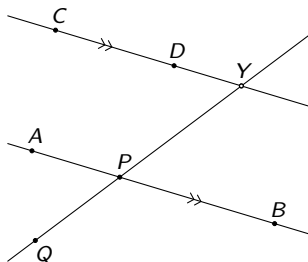
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Proclus' axiom

Proclus' axiom

Axiom (Proclus' axiom)

$$AB \parallel CD \wedge \text{Col } ABP \wedge \neg \text{Col } ABQ \Rightarrow \\ \exists Y, \text{Col } CDY \wedge \text{Col } PQY$$



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Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A-B-C \wedge A'-B'-C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A-B-C \wedge \neg B-C-A \wedge \neg C-A-B$
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Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A-B-C \wedge A'-B'-C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A-B-C \wedge \neg B-C-A \wedge \neg C-A-B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A-B-C \vee B-C-A \vee C-A-B$
Proclus	
Continuity	$\forall \exists \Upsilon, (\exists A, (\forall XY, \exists X \wedge \Upsilon Y \Rightarrow A-X-Y)) \Rightarrow$ $\exists B, (\forall XY, \exists X \wedge \Upsilon Y \Rightarrow X-B-Y \vee X = B \vee B = Y)$
Point equality decidability	$X = Y \vee X \neq Y$

Introduction

Tarski's System of Geometry

Gupta's and Szczerba's contributions

An independent version of Tarski's system of geometry?

Conclusion

Results

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- We had to correct one of Gupta's models.

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- An independent version of Tarski's system of geometry.
- We had to correct one of Gupta's models.
- Inner Pasch holds in the models that were given for outer Pasch.

Thank you!