Dominions in Filtral Quasivarieties

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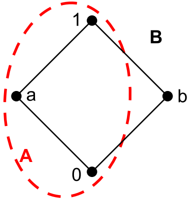
Example

Let $\mathscr{D}_{01}$ be the variety of bounded distributive lattices.



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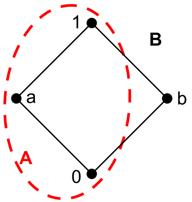

Observe that $b \in \mathrm{dom}^{\mathbf{B}}_{\mathscr{D}_{01}} \mathbf{A}$.

Dominions in $\mathscr{D}_{01}$

Let $\mathscr{B}$ be the class of boolean algebras. Given $\mathbf{A} \in \mathscr{D}_{01}$, there is a unique $\bar{\mathbf{A}} \in \mathscr{B}$ such that:

- $\mathbf{A} \leq \bar{\mathbf{A}}\big|_{\mathscr{D}_{01}}$
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Theorem (Wasserman 2001)

Let $\mathbf{A} \leq \mathbf{B} \in \mathscr{D}_{01}$, and let $b \in B$. T.f.a.e.:

- $b \in \mathrm{dom}^{\mathbf{B}}_{\mathscr{D}_{01}} \mathbf{A}$
- *b is in the subalgebra of $\bar{\mathbf{B}}$ generated by A.*

Discriminator varieties

The (quaternary) discriminator on a set A is the function $\mathrm{d}_A : A^4 \to A$ defined by

$$\mathrm{d}_A(x, y, z, w) = \begin{cases} z & \text{if } x = y \\ w & \text{if } x \neq y \end{cases}$$

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Examples:

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- Varieties of rings generated by finitely many finite fields,
- Monadic algebras.

Filtral quasivarieties

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- *For each $\mathbf{A} \in \mathscr{M}$ we have*
  $\mathbf{A} \vDash \alpha(x,y,z,w,u) \leftrightarrow \mathrm{d}_A(x,y,z,w) = u.$

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Suppose $\mathscr{M}$ has the AP and $\mathscr{M}_{\mathrm{ec}}$ is axiomatizable. Let $\mathbf{A} \leq \mathbf{B} \in \mathscr{Q}$ and $b \in B$, the following are equivalent:

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- There are a conjunction of equations $\delta(\bar{x}, y)$ and $\bar{a} \in A$ such that:
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    - $\mathscr{M}_{\text{ec}} \vDash \forall \bar{x} \exists ! y\, \delta(\bar{x}, y)$.

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$$\Delta := \{\delta(\bar{x}, y) : \mathscr{M}_{ec} \models \forall \bar{x} \exists! y \, \delta(\bar{x}, y)\}.$$

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For each $\delta(x_1, \ldots, x_n, y) \in \Delta$ let f_δ be a new n-ary function symbol and let

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- $\mathrm{dom}_{\mathbf{B}}^{\mathscr{Q}} \mathbf{A} = \langle A \rangle^{\bar{\mathbf{B}}} \cap B$ *for all* $\mathbf{A} \leq \mathbf{B} \in \mathscr{Q}$.

Thank you!